Notes on indefinite Metric Theory of Gupta & Bleuler 1

we start from commutation relations

$$[a_\mu(k), a_\nu^+(k')] = -\delta_{\mu\nu}\,\delta(k_2 - k_2')$$

So for $\mu = 1, 2, 3$, e.g. $[a_1, a_1^+] = 1$ (1)

but $[a_4, a_4^+] = -1$ (2)

Hamiltonian takes the form

$$H = \sum_k A_\mu^+(k) A_\mu(k) = \sum_k A_1^+A_1 + A_2^+A_2 + A_3^+A_3 - A_4^+A_4$$

$$= \sum_k (n_1 + n_2 + n_3 - n_4)$$

where $n_1 = A_1^+A_1$, etc and $n_4 = A_4^+A_4$ as well.

In ordinary treatment we treat (2) by assuming a_4 is creation operator, a_4^+ destruction.

We now try to keep assumption that a_4^+ is creation operator, and consider base states

$$\frac{1}{n!}\, a_4^+(k_1)\, a_4^+(k_2) \ldots a_4^+(k_n)\, |\Psi_0\rangle$$

Consider $|\Psi_1\rangle = a_4^+(k_1)\,|\Psi_0\rangle$

($\frac{1}{n!}$ for $k_1 = k_2 = k_n$ at any rate)

Then $\langle \psi_1 | \psi_1 \rangle = \langle \Psi_0 | a_4 \, a_4^{+} | \Psi_0 \rangle$

$= \langle \Psi_0 | -1 + a_4^{+} a_4 | \Psi_0 \rangle$

$= -1$ if $\langle \Psi_0 | \Psi_0 \rangle = 1$

and $a_4 | \Psi_0 \rangle = 0$

Similarly $\langle \psi_2 | \psi_2 \rangle = \frac{1}{2} \langle \Psi_0 | a_4 \, a_4 \, a_4^{+} a_4^{+} | \Psi_0 \rangle$

$= \frac{1}{2} \langle \Psi_0 | a_4 (-1 + a_4^{+} a_4) a_4^{+} | \Psi_0 \rangle$

$= \frac{1}{2} \langle \Psi_0 | -1 (-1 + a_4^{+} a_4) + a_4 a_4^{+} (-1 + a_4^{+} a_4) | \Psi_0 \rangle$

$= \frac{1}{2} \langle \Psi_0 | 1 + 1 | \Psi_0 \rangle = 1$

In general $\langle \psi_n | \psi_n \rangle = (-1)^n$.

So basis states have negative norm for odd n.

Hence if $\psi = \sum a_n | \psi_n \rangle$

$\langle \psi | \psi \rangle = \sum a_n^{*} a_n \langle \psi_n | \psi_n \rangle$

$= \sum (-1)^n a_n^{*} a_n$, which is

not positive definite, but corresponds to an indefinite metric in the Hilbert space

Then for metric $g_{\mu\nu}$, for orbitals ψ

$$\langle \psi | \psi \rangle = \sum g_{\mu\nu} a_\mu^* a_\nu \, , \text{ where}$$

$g_{\mu\nu}$ is metric tensor $\langle \psi_\mu | \psi_\nu \rangle$ and this form is positive definite in ordinary Hilbert space of Q. Mechanics.

We now admit possibility of negative norms

Eigenvalues of $n_\psi = a_\psi^\dagger a_\psi$

Consider $|\psi_1\rangle = a_\psi^\dagger |\psi_0\rangle$

Then
$$n_\psi |\psi_1\rangle = a_\psi^\dagger a_\psi a_\psi^\dagger |\psi_0\rangle$$
$$= a_\psi^\dagger \left[-1 + a_\psi^\dagger a_\psi \right] |\psi_0\rangle$$
$$= -1 \times a_\psi^\dagger |\psi_0\rangle$$

Similarly
$$n_\psi |\psi_n\rangle = a_\psi^\dagger a_\psi \, . \, \frac{\overbrace{a_\psi^\dagger a_\psi^\dagger a_\psi^\dagger \cdots}^{n \text{ factors}}}{n!} |\psi_0\rangle$$
$$= \left(a_\psi^\dagger (-1) \frac{\overbrace{a_\psi^\dagger \cdots a_\psi^\dagger}^{n-1 \text{ factors}}}{n!} + \frac{a_\psi^\dagger a_\psi^\dagger a_\psi a_\psi^\dagger}{n!} \cdots \right) |\psi_0\rangle$$
$$\to -n |\psi_n\rangle$$

Hence n_ψ has eigenvalues $-n_\psi$

Note also $\langle \psi_L | H | \psi_L \rangle$ for $n_1 = n_2 = 0$

$\rightarrow 0$ Since $H \sum_r ({}^nC_r)^{\frac{1}{2}} |n-r, r\rangle$

$= n \sum ({}^nC_r)^{\frac{1}{2}} |n-r, r\rangle$

$\therefore \langle \,|H|\, \rangle = n \sum_r^n (-1)^{r}\, {}^nC_r = 0.$

and hence eigenvalues of Hamiltonian are positive definite.

Also a_4 & $a_4^\dagger$ can be represented in terms of the $|\psi_n\rangle$ by matrices as shown in Jauch & Rohrlich, p.

thus $a_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{pmatrix}$

[so a_4 is really an annihilation operator in effect.] → $a_4^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$

Subsidiary condition now reads $(a_3 - a_4)|\Psi\rangle = 0$ corresponding to $\frac{\partial A_\mu^{(+)}}{\partial x}|\Psi\rangle = 0$

where

$|\Psi\rangle = |\Psi_0\rangle + \sum_{n=1}^{\infty} c_n |\psi_n\rangle$ c_n are arbitrary coeffs.

where in its final form

$|\psi_n\rangle = \sum_r \binom{n}{r}^{\frac{1}{2}} |n-r, r\rangle$ for $n_3 + n_4 = n$ ($\binom{n}{r}$ is binomial coeff.)

Hence $\langle\Psi|\Psi\rangle = \langle\Psi_0|\Psi_0\rangle = 1$ for all Ψ's of this form (known as the Gupta-Bleuler set L_4) — Different admixtures of $|\psi_n\rangle$, correspond to different gauges, give no observable result, so we can take

Vacuum condition as ~~Ψ~~ $a_3 |\Psi_0\rangle = 0$
$a_4 |\Psi_0\rangle = 0$.

Expectation values of operators are formed between the base vectors ψ_n with negative norms for odd ψ, in ordinary way.

Only new departure in formalism is that we must not assume $\langle\psi_n|\psi_n\rangle = 1$ is possible for all base vectors, i.e. we are not dealing with an ordinary Hilbert space.

We note as usual that if $\frac{\partial A_\mu^+}{\partial \mu} |\Psi_0\rangle = 0$ then full Lorentz condition is satisfied for expectation values, viz

$$\langle \Psi_0 | \frac{\partial A_\mu^+}{\partial \mu} + \frac{\partial A_\mu^-}{\partial \mu} | \Psi_0 \rangle$$
$$= \langle \Psi_0 | \frac{\partial A_\mu^+}{\partial \mu} | \Psi_0 \rangle + \langle \frac{\partial A_\mu^+}{\partial \mu} \cdot \Psi_0 | \Psi_0 \rangle$$

and both terms are then zero.

Notice that full length condition $(L^+ + L^-)|\Psi_0\rangle = 0$

requires both $(a_3 - a_4)|\Psi_0\rangle = 0$
and $(a_3^+ - a_4^+)|\Psi_0\rangle = 0$

as contrasted with the Gupta requirement

of only $(a_3 - a_4)|\Psi_0\rangle = 0$

Note! $\Psi_\beta^{in}\rangle$ is correct
Heisenberg state vector for
scattering from the state Φ_β $\Rightarrow$

Notes on Gasiorowicz's Elementary Particle Physics

S-Matrix introduced in Heisenberg representation

Note definition $S_{\alpha\beta} = \langle \psi_\alpha^{out} | \psi_\beta^{in} \rangle$

this corresponds to $|\psi_\beta^{in}\rangle = \sum_\alpha S_{\alpha\beta} |\psi_\alpha^{out}\rangle$

$|\psi^{out}\rangle$ is ~~an~~ eigenket of $H_0(+\infty)$

$|\psi^{in}\rangle$ is " $H_0(-\infty)$ and are actually eigenstates of H (total energy) cf. Schwinger.

Define operator $\underline{S}$ by $|\psi_\alpha^{in}\rangle = S|\psi_\alpha^{out}\rangle$

then $S_{\alpha\beta} = \langle \psi_\alpha^{in} | \underline{S} | \psi_\beta^{in} \rangle$

$= \langle \psi_\alpha^{out} | \underline{S} | \psi_\beta^{out} \rangle$

$= \langle \phi_\alpha | S | \phi_\beta \rangle$

where ϕ_α is eigenstate of H_0 Schrödinger i.e. of $H_0(0)$

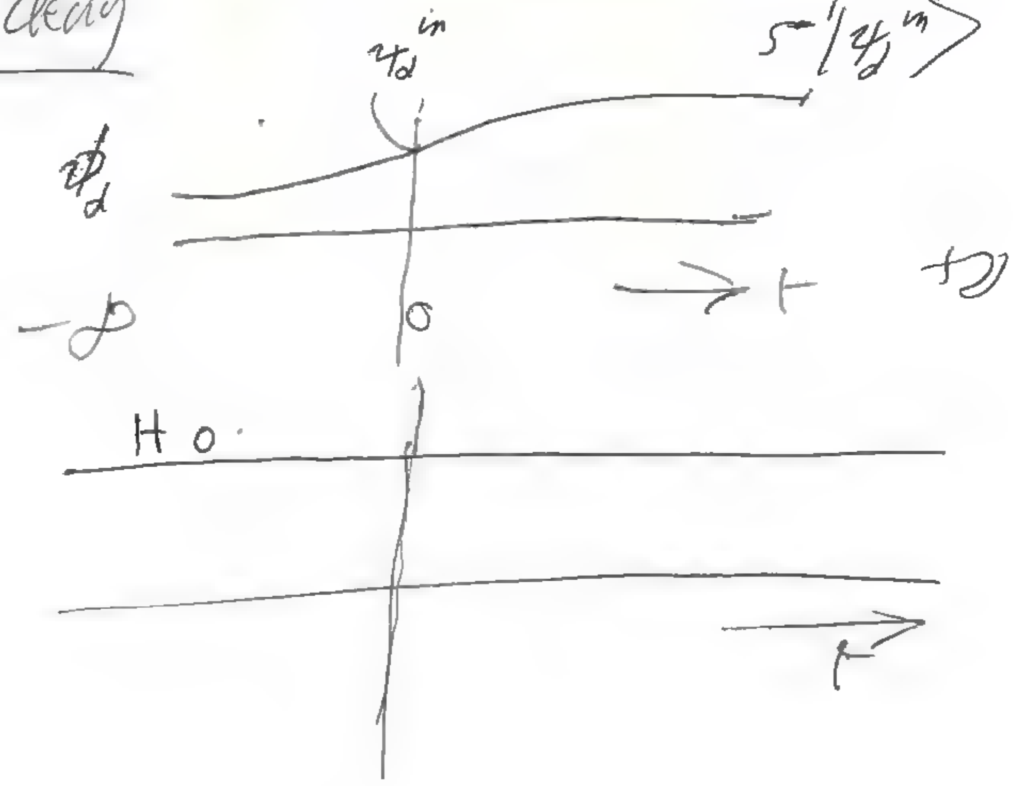
& $S = U(\infty, -\infty)$ is usual S-order definition.

We can then prove that $\underline{S} = S$, from relation

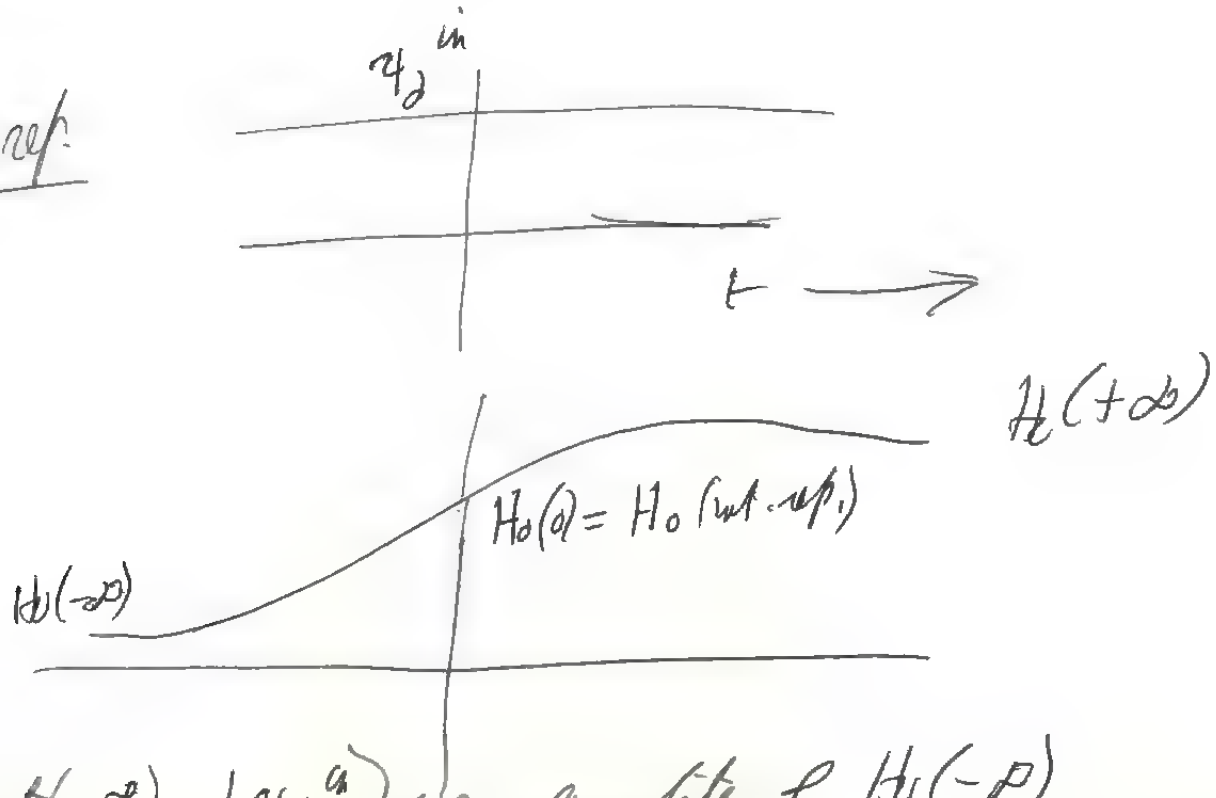
between ψ_α^{in} & ϕ_α e.g. $|\psi_\alpha^{in}\rangle = U(0,-\infty)|\phi_\alpha\rangle$

Relationship between I.P. & Heisenberg rep. for scattering theory

Int. Rep.



Heisenberg rep.



clearly at $(-\infty)$ $|\psi_\alpha^{in}\rangle$ is eigenstate of $H(-\infty)$
just as $|\Phi_\alpha\rangle$ is eigenstate of H_0

Reduction formulas LSZ for scattering from certain 2-particle state to final state α

$$S_{\alpha;pq} = \langle \psi_\alpha^{out} | \psi_{pq}^{in} \rangle$$

$$= \lim_{t \to -\infty} \langle \psi_\alpha^{out} | A^+(p,t) | \psi_q \rangle$$

$$= \lim_{t \to -\infty} \langle \psi_\alpha^{out} | i \int_{x_0 = t} d^3x \ \phi^*(x) \overleftrightarrow{\partial^0} f_p(x) | \psi_q \rangle$$

now write $\int_{x_0=-\infty} d^3x = \int_{x_0=\infty} d^3x - \int dx \frac{\partial}{\partial x_0} [\ ]$

1st term yields $\langle \psi_\alpha^{out} | \psi_{pq}^{out} \rangle = S_{\alpha,pq} = (1)_{\alpha,pq}$.

where $(S-1)_{\alpha,pq} = i \int d^4x \langle \psi_\alpha^{out} | J^*(x) | \psi_q \rangle f_p(x)$

where $(\Box^2 + \mu^2) \phi^\alpha(x) = J^\alpha(x)$ is current source of meson field. We can then go from $J^\alpha(x)$ $\to J^\alpha(0)$ and perform the spatial integration

thus $J(x) = e^{iP_\mu x_\mu} J(0) e^{-iP_\mu x_\mu}$

so $\langle \psi_\alpha^{out} | J^\alpha(x) | \psi_q \rangle f_p(x) \longrightarrow \langle \psi_\alpha^{out} | J^\alpha(0) | \psi_q \rangle \times \int e^{i(P_\alpha - P_q - P_p)\cdot x} d^4x$

$$= \delta^4(P_f - P_i) \langle \psi_\alpha^{out} | J^*(0) | \psi_q \rangle$$

or in G's notation $\delta^4(P_f - P_i) \left(\psi_\alpha^{out}, J^\alpha(0) \psi_q \right)$

Where we have gone from ψ_0 to ϕ_0

By result $|\psi_0\rangle = c\, U(0,-\infty)\,|\phi_0\rangle$ Since $\langle\psi_0|\psi_0\rangle = 1$

so $c = \dfrac{1}{\langle\psi_0|U(0,-\infty)\,|\phi_0\rangle}$

$\langle\psi_0|U(0,-\infty)|\psi_0\rangle$ $\langle\phi_0|U^{+}(-\infty,0)|\phi_0\rangle$

order than $\langle\phi_0|U(\infty,0)|\psi_0\rangle \rightarrow \langle\phi_0|U(0,-\infty)|\phi_0\rangle$

By repeated use we can ~~extract~~ "contract" other field operators ?

until we are left. a matrix element between $\langle \Psi_0 |$, $| \Psi_0 \rangle$

In perturbation theory we have to reduce

expressions of the form $\langle \Psi_0 , \hat{A}(x_1) \ldots \hat{A}(x_n) \Psi_0 \rangle$

$\hat{A}$ satisfy ordinary canonical commutation relations.

we express $\hat{A}$ in terms of $A^{(0)}$ the solution

op. solution at $t=0$.

we write ~~$\hat{A}(x) = U(t,0) A^0(x)$~~ U

$$A^0(x) = U(t,0)\hat{A}(x)\, U^{-1}(t,0)$$

$$= U(t,0)\hat{A}(x)\, U^{+}(t,0)$$

Hence $\hat{A}(x) = U^{+}(t,0) A^0(x)\, U(t,0)$ $\quad U(t_1,0)\, U^{-1}(t_2,0) = U(t_1,0)U(0,t_2) = U(t_1,t_2)$

then $\langle \Psi_0 | \hat{A}(x_1) \ldots \hat{A}(x_n) | \Psi_0 \rangle = \langle \Psi_0 , U^{+}(t_1,0) A^0(x_1)\, U(t_1,0)\, U^{+}(t_2,0)$

$A^0(t_2)\, U(t_2,0) \ldots \Psi_0 \rangle = (\Psi_0, U^{+}(t_1,0) A^0(t_1) U(t_1,t_2)$

$A^{(0)}(t_2) \ldots U(t_n,0) | \Psi_0 \rangle$.

So $\langle \Psi_0, T(\hat{A}(x_1) \ldots \hat{A}(x_n)) \Psi_0 \rangle = \lim_{\substack{t' \to \infty \\ t'' \to -\infty}} \dfrac{\langle \Phi_0 , T(U(t',t'') A^0(x_1) \ldots A^0(x_n)) | \Phi_0 \rangle}{(\Phi_0, U(t',t'') \Phi_0)}$

Strangeness — we have. for pions $Q = T_3$

for baryons $Q = \frac{1}{2} N_B + T_3$

N_B is baryon no.

In general relation between charge Q & T_3 is

$$\boxed{Q = \frac{1}{2} N_B + \frac{1}{2} S + T_3}$$ (Gell-Mann 1953)

$S = 0$ for pions & nucleons.

$S = -1$ for Λ^0

K meson produced in association with Λ^0 must have $S = 1$ (is for K^+ & K_0 $S = 1$; for K^- & $\bar{K}_0$ $S = -1$)

(note reaction $n + p \to \Lambda^0 + \Lambda^0 + \pi^+$ is forbidden)

for Σ's $S = -1$

Hypercharge Y is defined by $\boxed{Y = (N_B + S)}$

so $Q = Y + T_3$

Ξ's have $S = -2$

$\frac{1}{2}\tau_i$ are generators of the infinitesimal group.

τ_i are the usual Pauli Spin matrices. [illegible]

i in $e^{i\tau_i \cdot \theta_i}$ makes transformation unitary as the τ's are Hermitian.

Unitary symmetry

1

· starts with isotopic spin formalism

$e^{i\alpha_i\tau_i}$ yields rotation through components $\alpha_i = n_i\theta$

θ is angle of rotation

n_i is cosines unit vector along axis of rotation.

Infinitesimal transformation is

$1 + \frac{1}{2}i\, d\alpha\, \tau$ → 2x2 unitary unimodular transformation

generalization → 3x3 unitary unimodular = $(2 \times 9) - (9+1)$

= 8 ~~contin~~ parameter group

infinitesimal generator is $1 + i\sum_{i=1}^{8} \alpha_i F_i$

then F_i satisfy commutation relations $[F_i, F_j] = i f_{ijk} F_k$.

SU3 is a Lie group of rank 2 — two commuting generators — can be diagonalized simultaneously

– Take T_3 and Y for these generators.

Take states which support irreducible rep. – labelled by eigenvalues of T_3 & Y in 2-dimensional plot

shift operators change these eigenvalues.

– from maximum values of T_3 & Y we can work out set of contiguous states – traces out hexagonal boundary + interior points

Degenerate Jun is a Δ — gets 3 representation. ?

[illegible] [illegible] reduction of Tensor product of

irreducible reps $8 \times 8 = 1 + \cancel{3 + 3^{\alpha}} + 10^{*} + 27$
$\qquad\qquad\qquad 8 + 8 + 10 +$

Reduction coefficients are Clebsch-Gordan coefficients.

[Note on rotations

$A_{\mu\nu} \simeq \delta_{\mu\nu} + \varepsilon_{\mu\nu k}\, n_k \,.\, \theta$ small θ

[illegible] $x'_\mu = A_{\mu\nu}\, x_\nu = x_\mu + \varepsilon_{\mu\nu k}\, x_\nu n_k \,.\, \theta$ ①

$= x_\mu + (\hat{n} + \hat{n}\theta)$

we can write this as

$A_{\mu\nu} \simeq \delta_{\mu\nu} + \omega_{\mu\nu}$ $\qquad \omega_{\mu\nu} = \varepsilon_{\mu\nu k}\, n_k\, \theta$

for rotation $\omega_{\mu\nu}$ is $\qquad$ so $\omega_{12} = -\omega_{21} = n_3\, \theta$

antisymmetric, which is what ① states $\qquad$ 2 [illegible].

For rep. of rotation group.

$M \longrightarrow R(\hat{n}\, \theta)$

for infinitesimal element $\quad M = 1 + \dfrac{\partial M}{\partial \omega_{\mu\nu}}\, \omega_{\mu\nu}$

$M_{ij} = 1 + S^{ij}_{\mu\nu}\, \omega_{\mu\nu}$

$= 1 + S^{ij}_{r=1,2,3}\, n_r\, \theta$

$= 1 + S^{ij}_{r}\, \varepsilon_r \qquad \varepsilon_r = n_r\, \theta$]

Invariant coupling in 8-fold way

is generalization of $\bar{\psi}_a (T_i)_{ab} \psi_b \phi_i$

$\longrightarrow \quad \bar{\psi}_m (F_i)_{mn} \psi_n \phi_i = -i f_{imn} \bar{\psi}_m \psi_n \phi_i$

using representation $(F_i)_{mn} = -i f_{imn}$.

But $f_{imn} = \frac{1}{4i} Tr([\lambda_m, \lambda_n] \lambda_i)$

So coupling takes form $-\frac{1}{4} Tr([\lambda_m \bar{\psi}_m, \lambda_n \psi_n] \lambda_i \phi_i)$

$\bar{B} = \frac{1}{\sqrt{2}} \lambda_m \bar{\psi}_m \qquad \cancel{B = \frac{1}{\sqrt{2}}} \lambda_i \qquad Tr([\bar{B}, B] M)$

$B = \frac{1}{\sqrt{2}} \lambda_n \psi_n \qquad M = \frac{1}{\sqrt{2}} \lambda_i \phi_i$

8 components of ψ

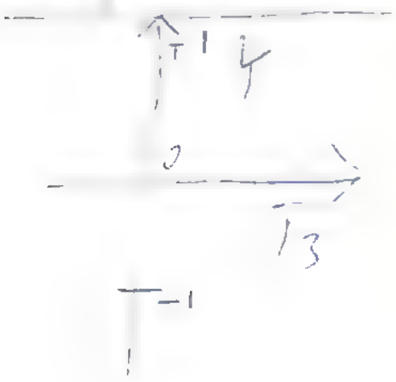
$\partial \quad \phi_i$

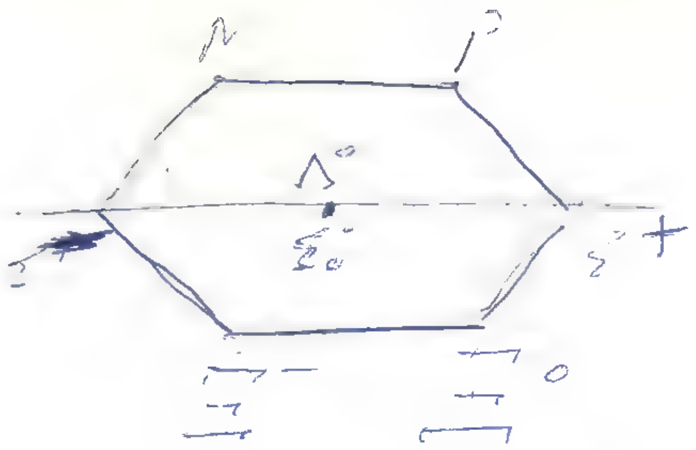
2nd type of invariant coupling is $Tr(\{\bar{B}, B\} M)$

General 4 invariant coupling is written as

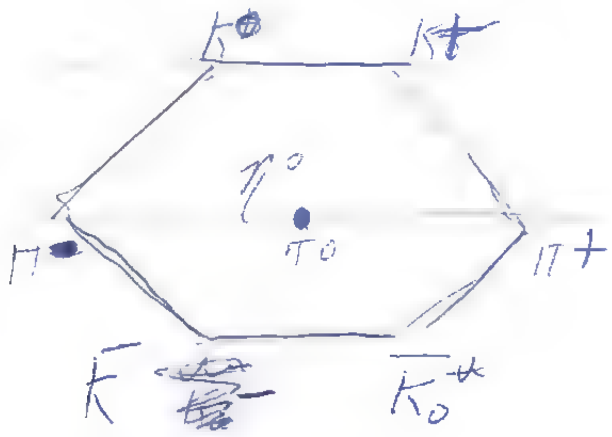
$$-(1-\alpha) \; tr([\bar{B}, B] M) + \alpha \, Tr(\{\bar{B}, B\}, M)$$

Baryon octet



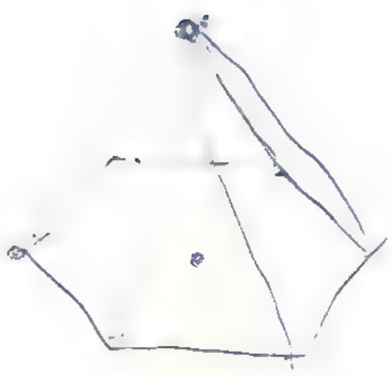


Meson octet



Two doublets 4
Triplet 3
singlet 1

$$\overline{8}$$ particles in all



Connection between state & field ~~selections~~ version of higher symmetry schemes

If $U \phi_\alpha U^{-1} = F_{\alpha\beta} \phi_\beta$ for transformation & field U

then single particle states $\phi_\alpha \Psi_0$ also support the rep. $F_{\alpha\beta}$ — i.e.

$$\cancel{U \phi_\alpha U^{-1}} \quad U \phi_\alpha |\Psi_0\rangle = U \phi_\alpha U^{-1} U |\Psi_0\rangle$$
$$= F_{\alpha\beta} \phi_\beta |\Psi_0\rangle \qquad (1)$$

so the states $\phi_\alpha |\Psi_0\rangle$ support the rep.

conversely if $\phi_\alpha |\Psi_0\rangle$ supports a rep.

(1) leads to $U \phi_\alpha U^{-1} = F_{\alpha\beta} \phi_\beta$, assuming always that $U |\Psi_0\rangle = 0$

Note on Infinitesimal Generators of Symmetry Transformations

$U = 1 + \epsilon K \approx e^{iK}$ in gen.

K is generator

then $i[K, \phi]$ gives change in field ϕ which must satisfy commutation relations & field equations again $(\phi' = \phi + i[K, \phi])$

We also require Schrodinger Eq. to be unchanged in form. Hence $[K, H] = 0$

In S. representation, this leads to eigenvalues of H satisfying invariance reqs for K as in N.R. theory (or infinitesimal S. or in H.P. eq. being covariant in Wigner's treatment e.g.)

If we deal in interaction rep. we require $[K, H'] = 0$, H' is interaction total energy — this is equivalent to $[K, S] = 0$ in Tomonaga

[illegible]'s treatment.

Notice that $[K, H] = 0$ is not required for field equations to be covariant — this follows from covariance of U.

But $[K, H] = 0$ implies K is constant in time and can be identified with constants of motion from Noether's theorem, using commutation relations, which is how we do derive from invariance of field equations via. from invariance of the Lagrangian density — this difference often to be overlooked.

Every [illegible] C [illegible] must commute [illegible] in the Int. rep. the S representation. — $[K, H]$ is equivalent to covariant Tomonaga-Schwinger equation p. [illegible]

presumably.

$H'(t)$ being a scalar, this amounts to $[K, H'] = 0$, which we write it out.

Baryon resonances

Dalitz plot 3 particle reactions such as

$K^- + p \to \Lambda^0 + \pi^+ + \pi^-$ plot $(\omega_2+\omega_3)^2 - (\underline{p}_2+\underline{p}_3)^2$

against $(\omega_1+\omega_2)^2 - (\underline{p}_1+\underline{p}_2)^2$ e.g. peaks at M^2

if unstable particle of mass M is produced.

note $(\omega_2+\omega_3)^2 - (p_2+p_3)^2 \propto \omega_1$ — energy of 3rd

particle, so Dalitz plot really shows events as

a function of energies of two of the emerging particles.

1st ~~Boson~~ Baryon resonance to be discovered

Resonance occurs in $\pi^+ + p \to \pi^+ + p$. (also in $\pi^- + p$)

a number of peaks occurs. at 1238 MeV

1518 MeV, 1688 MeV 1922 MeV & 2200 MeV.

all have $Y=1$ (hypercharge, $S=0$)

Also strange resonances, $Y=0$ & $Y=-1$ are occur.

Discovery of the Ω^- as i-spin singlet of negative charge

and at $Y=-2$, $S=-3$ gives $Q = \frac{1}{2} - \frac{3}{2} = -1$

predicted as tenth member of a 10 (0,3) representation

of SU3.

Meson resonances

(indirect process)

ρ meson observed. $\pi^- + p = \pi^+ + \pi^- + n$. (π^+, π^-) peak.

ω-meson observed in $p + \bar{p} = \pi^+ + \pi^+ + \pi^0 + \pi^- + \pi^-$

sharp peak for $(\pi^+ \pi^0 \pi^-)$

ϕ-meson in $K\bar{K}$ channel.

Decay of ρ meson is allowed by strong interaction

B-meson or Buddha

in the reaction $\pi^+ + p \to \pi^0 + \pi^+ + \pi^- + \pi^+ + p$.

a four particle peak was found ~~correlating~~

$\pi^+ \pi^0 \pi^-$ strictly correlated at the ω-meson mass

Buddha is interpreted as an $(\omega\pi)$ resonance.

Weak Interactions:

form of the β - interaction, non-conservation of parity from experiments of Wu's Ambler on correlation between p_e & σ for decay of polarized nucleus. $p_e \cdot \sigma$ is a pseudoscalar quantity

If initial state is of even parity, final state is $|\Psi_f\rangle = f\,|\Psi_{odd}\rangle + (1-f)\,|\Psi_{even}\rangle$ say.

then
$$\langle p_e \cdot \sigma \rangle_f = \frac{2\,\mathrm{Re}\, f(1-f)\,\langle \Psi_{odd} | p_e \cdot \sigma | \Psi_{even}\rangle}{f^2 + (1-f)^2}$$

$$\propto \frac{f(1-f)}{f^2+(1-f)^2}$$ is zero if $f = 0$ or 1

i.e. non-vanishing $\langle O_{pseudoscalar}\rangle$ means mixture of even & odd parity states is required

Note that $\langle \Psi_{even} | O_{pseudoscalar} | \Psi_{odd}\rangle$ is non-vanishing

But $\langle \Psi_{even} | O_{pseudoscalar} | \Psi_{even}\rangle$ e.g. vanishes

$= \langle \Psi_{even} | P^{-1}\, \underline{P O P^{-1}}\, P | \Psi_{even}\rangle$

$= - \langle \Psi_{even} | O | \Psi_{even}\rangle$ where $\langle \Psi_{even} | O | \Psi_{even}\rangle = 0$

Since $P^2 = 1$ $\quad P^{-1} = P$ $\quad$ and $P = P^{*}$ unitary operator $\quad$ the result follows

Notice that P_x is now a field momentum operator etc

So $P\, Q(x)\, P^{-1} = -\, Q(-x)$ has in fact to be integrated over all x and $\int d^3x\, Q(-x)$

$$= \int d^3x\, Q(x)$$

and changing $x \to -x'$

Universal Fermi Interaction explains $\pi^- \to \bar{e} + \bar{\nu}$ e.g.

as $\pi^- \to \bar{p} + n \to \bar{e}\, \bar{\nu}$

Proposed form is $H_W = \frac{G}{\sqrt{2}} \int d^3x\, J_\alpha(x)\, J^{\alpha\,+}(x)$

with $J_\alpha(x) = \bar{\psi}_e \gamma_\alpha (1-\gamma_5) \psi_\nu + \bar{\psi}_\mu \gamma_\alpha (1-\gamma_5) \psi_\nu$
$+ \bar{\psi}_n \gamma_\alpha (1-\gamma_5) \psi_p + \cdots$

Could be interpreted as due to a heavy intermediate boson $m_W \geq 2$ BeV.

We can also write $J^\alpha(x) = J_\ell^\alpha(x) + J^\alpha_{\text{strong}}(x)$

i.e. a leptonic current $\bar{\psi}_e \gamma^\alpha (1-\gamma_5) \psi_{\nu_e} + \bar{\psi}_\mu \gamma^\alpha (1-\gamma_5) \psi_{\nu_\mu}$.

$J^\alpha_{\text{strong}} = J^\alpha_{V,0} + J^\alpha_{A,0} = \bar{\psi}_n(x) \gamma^\alpha (1-\gamma_5) \psi_p(x)$

$J_{V,0}$ is vector current, $\Delta S = 0$

$J_{A,0}$ is axial vector current $\Delta S = 0$

More generally with e.m. current.

$$J^{\alpha}_{\text{strg}} = J^{\alpha}_{V,0} + J^{\alpha}_{A,0} + J^{\alpha}_{V,1} + J^{\alpha}_{A,1}$$

where last two terms give $|\Delta S| = 1$

Decay of K_0 & $\bar{K}_0$ discussed by Gell-Mann & Pais in 1955.

eigenstates of mass operators are eigenfunctions of CP but not of S.

Introduce K_L & K_S as linear combination of K_0 & $\bar{K}_0$

then K_L & K_S are eigenstates of CP but not of S.

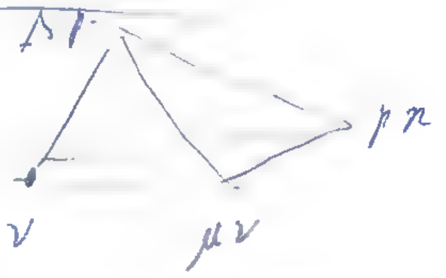
In decay S is not conserved but CP is

ex. K_0 produced by strong interaction is a mixture of K_L & K_S — as beam runs forward K_S decays leaving K_L only — now K_L is a mixture of K_0 & $\bar{K}_0$ so $\bar{K}_0$ is regenerated — if $\bar{K}_0$ is now absorbed by strong interactions, we are left with K_0 only which regenerates K_S so decay of K_S is again

observable. K_S decays into 2 pion state or 3 pion 4

K_L — — 3 pion decay. only

Gell-Mann Pais tetrahedron



coupling between Λ p vertex & other vertices involves a change in strangeness.

Decays can be leptonic or non-leptonic

remember. $\pi \equiv$ p n by strong interaction

and ~~π π~~ ≡ $K \equiv$ Λ p by strong interaction.

F_A also arises in ordinary β decay $p \to n + e + \nu$

where cross-section, including strong interactions

involves M.E.

$$\bar{u}(p_e)\gamma^\mu(1-\gamma_5)u(p_\nu)\,\bar{u}(p_n)\{\gamma^\mu F_1 - \gamma^\mu\gamma_5 F_A - \ldots\}u(p_p)$$

F_1 is form factor known from electron-nucleon scattering — then F_A is the axial form factor which means in fact a renormalization of the axial vector coupling constant due to strong interaction effects.

we write $J^{\mu}_{V,0} = \bar{\psi}_n \gamma^{\mu} \psi_p +$

$= \mathcal{J}^{(1)\mu} - i\, \mathcal{J}^{(2)\mu}$.

$\mathcal{J}$ is the isotopic spin current which is conserved.

Hence we assume $J_{V,0}$ is a conserved current. CVC hypothesis

PCAC means partially conserved axial current

assumption is $\partial_\alpha \mathcal{J}^{\alpha}_5 \propto$ meson field operator.

A_5 is the axial current $J^{(2)}_{A,0}$

— basis of Goldberger-Treiman relation

between $F_A(0)$ & pion decay rate

F_A is axial form factor — occurs in β decay of Λ^0

or $(\psi_p, J^{\alpha}_{V,1} + J^{\alpha}_{A,1}\, \psi_\Lambda) = \frac{1}{(2\pi)^3}\, \bar{u}(p_p)(F_V \gamma^\alpha - F_A \gamma^\alpha \gamma_5) u(p_\Lambda)$

$\longrightarrow F_\pi \simeq \frac{\sqrt{2}}{g} F_A(0)$

F_π is form factor governing decay of π-meson

We turn now to strangeness-changing currents

$J_{V,1}$ is not conserved in limit of all baryon masses equal.

Cabibbo suggested $J^{\alpha}_{V,1} = \mathcal{J}^{(4)\alpha} - i\,\mathcal{J}^{(5)\alpha}$

where $F^{(4)}$, $F^{(5)}$ are octet currents.

So in general Cabibbo wrote

$$J^{\alpha}_{V} = \underbrace{\cos\theta\left(\mathcal{J}^{(1)\alpha} - i\,\mathcal{J}^{(2)\alpha}\right)}_{\substack{\Delta S = 0 \\ \text{i-spin current}}} + \underbrace{\sin\theta\left(\mathcal{J}^{(4)\alpha} - i\,\mathcal{J}^{(5)\alpha}\right)}_{\Delta S = 1}$$

$$F^{(i)}(t) = \int_{x_0 = t} d^3x\, \mathcal{J}^{(i)\,0}(x) \quad \text{are generators of SU3}$$

axial currents obey

$$\left[\mathcal{J}_5^{(i)\alpha}, F^{(j)}\right] = i f_{ijk}\, \mathcal{J}_5^{(k)\alpha}(x)$$

i.e. have octet transformation properties.

Also assume $\partial_\alpha \mathcal{J}_5^{(i)\alpha}(x) \propto \Phi_i(x)$

PCAC hypothesis

For J_A^α Cabibbo wrote

$$J_A^\alpha = \cos\theta\,(\mathcal{F}_5^{(1)\alpha} - i\,\mathcal{F}_5^{(2)\alpha}) + \sin\theta\,(\mathcal{F}_5^{(4)\alpha} - i\,\mathcal{F}_5^{(5)\alpha})$$

Gell-Mann's Current Algebra

Define $F_5^{(i)}(t) = \int_{x_0 = t} d^3x \; \mathcal{F}_5^{(i)0}(x)$

then. $[F^{(i)}(t), F_5^{(j)}(t)] = i f_{ijk} F_5^{(k)}(t)$ (1)

since $F_5^{(i)}$ has octet transformation law.

We also have of course.

$[F^{(i)}(t), F^{(j)}(t)] = i f_{ijk} F^{(k)}(t)$ (2)

What can we say about $[F_5^{(i)}(t), F_5^{(j)}(t)]$

$= i f_{ijk} F^{(k)}(t)$

assumed by Gell Mann to close the algebra.

Leads to Adler-Weisberger relation

$$\frac{1}{F_A^2(0)} = 1 + \frac{2M^2}{\pi g^2}\int_0^\infty \frac{d\nu'}{\nu'}\left(\sigma^-_{tot}(\nu') - \sigma^+_{tot}(\nu')\right)$$

over-estimates one for non-leading terms

Analiticity of S-Matrix — Dispersion relation etc. 1

We start from unitarity of the S-matrix

$$S S^{+} = S^{+} S = 1$$

this expresses fact ~~$|\psi_\alpha^{out}\rangle = S|\psi_\alpha^{in}\rangle$~~

$$|\psi_\alpha^{in}\rangle = S|\psi_\alpha^{out}\rangle$$

then $\langle \psi_\alpha^{in} | \psi_\alpha^{in}\rangle = \langle \psi_\alpha^{out} | S^{+} S | \psi_\alpha^{out}\rangle$

or more generally $\langle \psi_\beta^{in} | \psi_\alpha^{in}\rangle = \langle \psi_\beta^{out} | S^{+} S | \psi_\alpha^{out}\rangle = \delta_{\alpha\beta}$

shows that $(S^{+}S)_{\beta\alpha} = \delta_{\beta\alpha}$ in the out operator.

hence $S^{+}S = 1$

We write $S = 1 + R$ ~~$1 + iR$~~, gives

$$(1+R^{+})(1+R) = 1$$

or $1 + R^{+}R + R + R^{+} = 1$

or $-R^{+}R = R + R^{+}$ ~~if $R \Rightarrow iR'$, then~~

$R^{+}R =$ ~~$i(R - R^{+}) = 2i \, \mathrm{Im}\, R'$~~

now write. $\left(\psi_{\beta}^{(-)}, R\, \psi_{\alpha}^{(+)}\right) = -(2\pi)^4 i\ \delta(P_\beta - P_\alpha)\ T(\beta, \alpha)$ 3

we find $i\left(T^*(p,q;p'q') - T(p'q',pq)\right)$

$$= -(2\pi)^4 \sum_n \delta(P_n - p - q)\ T^*(n, p'q')\ T(n, p, q).$$

for forward scattering $p' = p,\ q = q'$

$\Rightarrow 2\,\mathrm{Im}\ T(p,q,pq) \propto |T(n,p,q)|^2.$

leads to $\mathrm{Im}\ f(W,0) = \frac{q}{4\pi}\ \sigma_{tot}(W)$

optical theorem

The scattering amplitude

$$F(W,\theta,\phi) = -\frac{16\pi^5 M}{W}\ T(p'q' : p, q)$$

relates scattering amplitude to the T matrix

(Cross-section is given in terms of f by.

$$\frac{d\sigma_{el}}{d\Omega} = \sum_{spin} |F(W,\theta,\phi)|^2 \Big)$$

This is the so-called optical theorem

But we can also derive generalized unitarity relations from the reduction formulae.

Evaluate $(p'q'|R|p,q) + (p'q'|R^{+}|p\,q)$ $= (s$ ①

as a sum over states, reduces to optical theorem in s-channel, but now gives contributions in the t-channel, also one-particle poles.

In general unitarity seen in optical theorem obtains additional terms every time new physical process needs considered.

① can be written as $(p'q'|R|pq) + (pq|R|p'q')^{*}$

$$\propto i\left(T(p'q';pq) - T^{*}(p,q,p'q')\right)$$

which is then evaluated by reduction formulae.

probably new concept
to regard whether anti particle
or boson some annihilation or
antiboson particle : etc
cf. D. G. H. Calthorpe.

Crossing Symmetry

$$s = (p_1 + p_2)^2$$
$$t = (p_3 - p_1)^2$$
$$u = (p_4 - p_1)^2$$

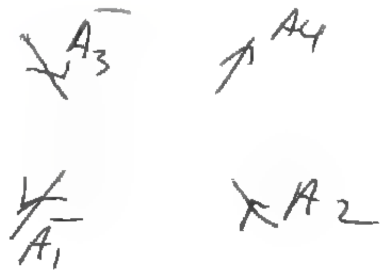
$A_1 + A_2 = A_3 + A_4$

mes nucl = mes + nucl.

s-channel $s > 0$
$t < 0$
$u < 0$

$\sqrt{s}$ = c.d.m. energy

u-channel cross 3 & 1 to give



$$A_2 + \bar{A}_3 = \bar{A}_1 + A_4$$

nucl. + anti meson = anti meson + nucleon.

$p_3 \Rightarrow -p_1$ $\quad s \rightarrow (p_2 - p_3)^2 = u$

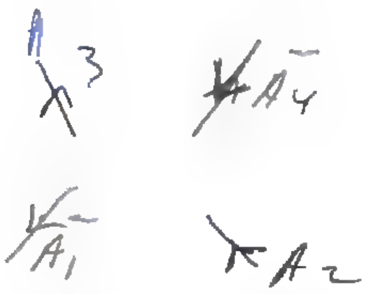
$p_1 \rightarrow -p_3$ $\quad t \rightarrow (p_3 - p_1)^2 = t$

$\quad u \rightarrow (p_4 + p_3)^2 = (p_1 + p_2)^2 = s$

now $\sqrt{u}$ = c.d.m. energy.

$u > 0, \; s < 0, \; t < 0$

t-channel cross 4 & 1



$$A_2 + \bar{A}_4 = \bar{A}_1 + A_3$$

nucl. + antinucl $\rightarrow$ meson + antimeson

$\sqrt{t}$ = c.d.m. energy

$p_4 \Rightarrow -p_1$ $\quad s \rightarrow (p_2 - p_4)^2 = t$

$p_1 \rightarrow -p_4$ $\quad t \rightarrow (p_3 + p_4)^2 = (p_1 + p_2)^2 = s$

$\quad u \rightarrow (p_4 - p_1)^2 = u$

So u-channel exchanges s and u, leaves t unchanged.
t-channel exchanges s & t, leaves u unchanged

[Notice we are assuming
that in crossed reactions
cross-sections are for $p_1 p_2 \to p_3, p_4$
the fact TCP theorem (which) refers
to reaction $p_3, p_4 \to p_1, p_2$

i.e. the replacements
$p_3 \to -p_3$
$p_4 \to -p_4$
$p_2 \to -p_2$
$p_1 \to -p_1$

which still leave s, t, u unchanged.

In general, PT invariance implies
$p_1, p_2 \to p_3, p_4$ has same x section
as $p_3, p_4 \to p_1, p_2$ so non-trivial is
same in this case as that discussed in affordable
page (cf Eden)]

T.C.P. inverse of $A_1 + A_2 = A_3 + A_4$

is $\bar{A}_3 + \bar{A}_4 = \bar{A}_1 + \bar{A}_2$ $\Bigg\}$ means $p_3 \to -p_1,\ p_4 \to -p_2$, $p_1 \to -p_3,\ p_2 \to -p_4$ $\Bigg\}$ $s \to s$, $t \to t$, $u \to u$ which is invariant

similarly T.C.P. inverse of u-channel is

$$A_1 + \bar{A}_4 = \bar{A}_2 + A_3$$

and T.C.P. inverse of t-channel is

$$A_1 + \bar{A}_3 = \bar{A}_2 + A_4$$

Because of T.C.P. theorem

for spinless bosons cross-sections for s, u, & t channels are all identical, leads to the notion of crossing symmetry in $f(s,t,u)$

---

we can use $-p_3$, $-p_4$ for 4-momenta of emerging particles, which gives symmetrical form

$$\left.\begin{aligned} t &= (p_1 + p_3)^2 \\ u &= (p_1 + p_4)^2 \\ s &= (p_1 + p_2)^2 \end{aligned}\right\} \qquad p_1 + p_2 = p_3 + p_4$$

Notice that $t + u + s = 2 + 2p_1 \cdot p_2 + 2 - 2\, p_1 \cdot p_3 + 2 - 2\, p_1 \cdot p_4$

$= \underline{s + t + u} = 6 + 2p_1 \cdot p_2 - 2p_1 \cdot (p_1 + p_2) = \underline{4 m^2}$

Dispersion relations

we consider two particle scattering as an example

$F(s,t)$ say describes $\langle p_3 p_4 | F | p_1, p_2 \rangle$

$s = (p_1 + p_2)^2 \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2.$

consider fixed value of t, now go to u-channel

thresholds will occur at $u = 4m^2 \quad 9m^2 \quad 16m^2 \quad \ldots$

corresponds to $s = 4m^2 - t_0 - u$

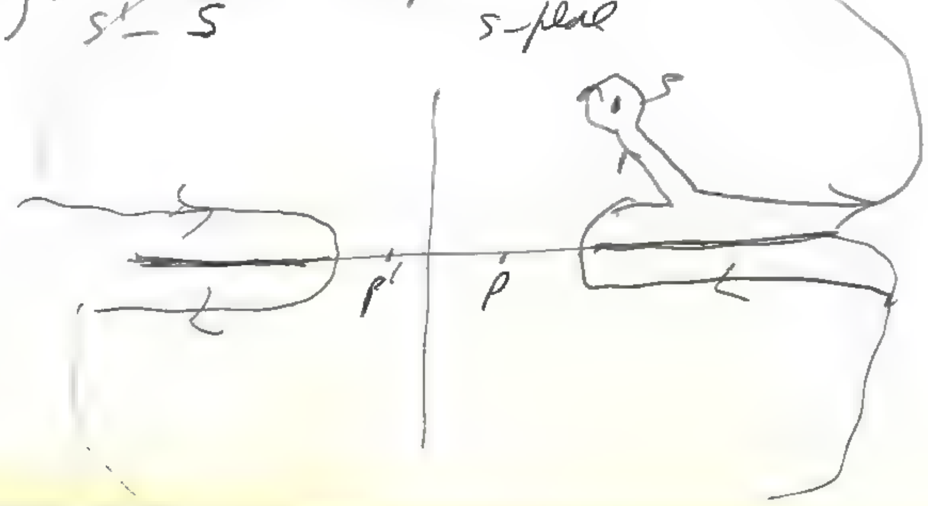
$= -t_0, \; -t_0 - 5m^2, \;$ so on.

So in complex s-plane, for fixed $t = t_0$, singularities are as shown.

poles at $s = m^2 \; (-t_0)$
and $s = 3m^2 - t_0$

To obtain dispersion relation for $F(s,t)$

consider $\frac{1}{2\pi i} \int \frac{F(s', t_0)}{s' - s}$ integrated round contour shown as C



sum is $- f(s, t_0) + \frac{1}{2\pi i}\int_{-t_0}^{-\infty} \frac{ds' \, F_u(s', t_0)}{s' - s}$

$$+ \frac{1}{2\pi i}\int_{4m^2}^{\infty} \frac{ds' \, F_s(s' t_0)}{s' - s} = \frac{2\pi i}{2\pi i}\sum_j \frac{A_j}{s_j - s}$$

where A_j is residue of $F(s', t_0)$ at poles s_j

Hence $$f(s, t_0) = \frac{1}{2\pi i}\int_{-t_0}^{-\infty} \frac{ds'}{s' - s} F_u(s', t_0) \quad (1)$$

$$+ \frac{1}{2\pi i}\int_{4m^2}^{\infty} \frac{ds'}{s' - s} F_s(s', t_0)$$

$$+ \underbrace{\sum_j \frac{A_j}{s - s_j}}_{= P}$$

pole terms P are given by

$$P = \frac{g_s^2}{s - m^2} + \frac{g_u^2}{s + t_0 - 3m^2}$$

g_s & g_t are constants.

F_s is discontinuity in F across R.H. cut. = disc. $F = 2i\,\mathrm{Im}\,F$

if F is real between cuts on real axis, so Schwarz reflection principle applies

$$\text{disc } F = F(x + i\varepsilon) - F(x - i\varepsilon) = F(x + i\varepsilon) - F^*(x + i\varepsilon)$$
$$= 2i\,\mathrm{Im}\,F$$

since F is $\lim_{\varepsilon \to 0} F(x + i\varepsilon)$ for physical amplitude in s-channel.

F_u is discontinuity in F in the u-channel.

Since in this case. Physical limit in u-channel.

is $\lim.\ F(s-i\varepsilon, t_0)$ for $s < 0$.

making use of ~~$s'+t'+$~~ $s' + u' + t = 4m^2$

we can write.

since $s + t_0 - 3m^2 = m^2 - u$

$$F(s,t,u) = \frac{g_s^2}{s-m^2} + \frac{g_u^2}{u-m^2} + \frac{1}{2\pi i}\int_{4m^2}^{\infty} \frac{ds' F_s(s',u',t)}{s'-s}$$

$$+ \frac{1}{2\pi i}\int_{4m^2}^{\infty} du' \frac{F_u(s',u',t)}{u'-u}. \quad ②$$

(g_u^2 has changed sign on 1st term)

Since $u' = 4m^2 - s' - t$ in 2nd integral

$ds' = -du'$, $s' = -t$, $u' = 4m^2$

$s' = -\infty$, $u' = \infty$.

$$s' - s = 4m^2 - u' - t - (4m^2 - u - t) = -(u' - u)$$

② gives relation to the Mandelstam rep. which takes the form

$$F(s,t,u) = P + \frac{1}{\pi^2}\iint \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} ds'dt'$$

$$+ \frac{1}{\pi^2}\iint \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} dt'du' + \frac{1}{\pi^2}\iint \frac{\rho_{us}(s',u')\, ds'du'}{(u'-u)(s'-s)} \quad ③$$

for double dispersion relation. Pure pole terms

Remember $s'+t'+u' = 4m^2$.

By integrating over t', we can reobtain single dispersion relation, if we remember $s+t+u=4m^2$ and keep t fixed in our derivation.

So s and u are related by $s = 4m^2 - t - u$.

To do this we can integrate 1st, 2nd terms in (3) directly and then analyze third term as follows $\frac{1}{\pi^2}\iint \frac{\rho_{us}(s',u')\,ds'du'}{(u'-u)(s'-s)}$.

Since

$$\frac{1}{(u'-u)(s'-s)} = -\frac{1}{(t'-t)(s'-s)} - \frac{1}{(t'-t)(u'-u)}$$

$$= \frac{-(u'-u)-(s'-s)}{(t'-t)(s'-s)(u'-u)} = \frac{-(u'+s')+(u+s)}{(\ )(\ )(\ )}$$

$$= \frac{-(4m^2-t')+(4m^2-t)}{(\ )(\ )(\ )}$$

$$= \frac{1}{(s'-s)(u'-u)} \text{ again}$$

$$\therefore \frac{1}{\pi^2}\iint = -\frac{1}{\pi^2}\iint du'ds' \frac{\rho_{us}(u',s')}{(t'-t)(s'-s)} - \frac{1}{\pi^2}\iint du'ds' \frac{\rho_{us}(u',s')}{(t'-t)(u'-u)}$$

One can now do the integrations to yield the single variable dispersion relations, such as (2)

e.g. F_s in (1) is given by

$$\frac{1}{2\pi i} F_s = \frac{1}{\pi^2}\int_{4m^2} dt' \, \frac{\rho_{st}(s', t')}{t' - t}$$

$$= \frac{1}{\pi^2}\int_{-\infty}^{-s'} dt' \, \frac{\rho_{us}(4m^2 - s' - t', s')}{t' - t}$$

minus sign differs from Eden?

where we have gone from u' to t' as variable of integration $\quad t' = 4m^2 - u' - s' \quad$ ~~$dt' = du'$~~ $du' = -dt'$

$u' = \infty$ goes to $t' = -\infty$.

$u' = 4m^2 \quad t' = -s'$

5

Subtractions we have assume $f(s, t_0) \to 0$ (a large (s)

a) integral over circle vanishes

If this is not the case, suppose

$$G(s, t_0) = \frac{F}{(s-s_1)\ldots(s-s_N)}$$ with $s_1 \ldots s_N$ are constant

satisfies this condition.

then diff dispersion relation to this ~~original~~ function we get

$$F(s, u, t_0) = \phi^{(N)} + \frac{g_s^2}{s-m^2} + \frac{g_u^2}{u-m^2} + \frac{(s-s_1)\ldots(s-s_N)}{2\pi i}$$

$$\times \left\{ \int_{4m^2}^{\infty} \frac{ds' F_s(s', u', t_0)}{(s'-s)(s'-s_1)\ldots(s'-s_N)} + \int_{4m^2}^{\infty} \frac{du' F_u(s', u', t_0)}{(u'-u)(u'-u_1)\ldots(u'-u_N)} \right\}$$

where ~~$t_1 \ldots t_N$~~ $u_1 \ldots u_N$ are defined by.

$$u_i = 4m^2 - t_0 - s_i)$$

and $s' + u' + t_0 = 4m^2$ as usual.

Φ^N is a polynomial of degree $(N-1)$ in s.

derived from ~~[illegible]~~ residues at poles $s_1 \ldots s_N$ — 6

then $$\frac{G(s')}{s'-s} = \frac{F(s')}{(s'-s)\cdot(s'-s_1)\cdots(s'-s_N)}$$

Residue at s_1 is $$\frac{F(s_1)}{(s_1-s)\,(s_1-s_2)\cdots(s_1-s_N)}$$

and $(s-s_1)\cdots(s-s_N) \times$ residue

$$= (s-s_2)\cdots(s-s_N) \times \text{const}$$

which is pol. of degree $(N-1)$ so

arbit. N coefficients of $\Phi^{(N)}$ are called <u>subtraction constants</u> & our disp. relation is said to have N subtractions.

In Mandelstam rep. $\rho_{st}(s', t')$ denotes double discontinuity across cuts in s & t channels simultaneously.

If $f(u)$ is not real for $u \leq a$

then $F_1(u) \equiv$ disc. F

$\equiv$ L.H.S. of unitarity condition

So that we can use the unitarity condition in its general form

in this case here.

When symmetry condition $(1|R|2) = (2|R|1)$ is satisfied

then disc. $= 2\,\mathrm{Im}$, and F is real on the real axes between the branch cuts

Simpler dispersion relations can be written

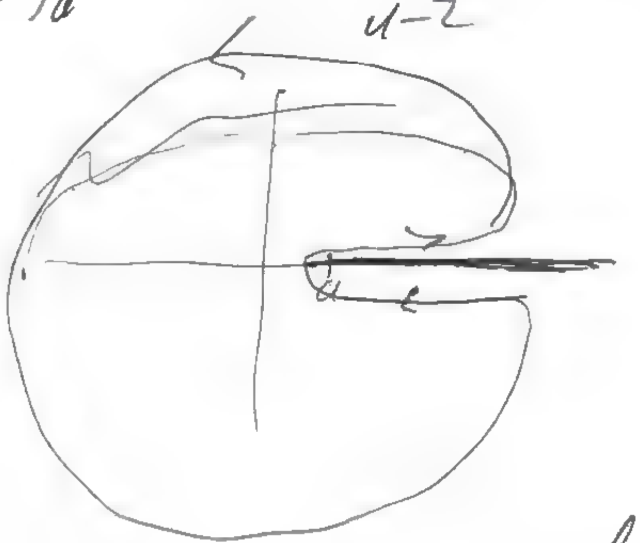
e.g. Eden (1967) gives simplest example.

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z}.$$

assume 1.) $f(w)$ no singularities except along axis real $> a$

2.) $f(w) \to 0$ as $w \to \infty$.

then $f(z) = \frac{1}{2\pi i}\int_a^\infty du\, \frac{f(u+i\varepsilon) - f(u-i\varepsilon)}{u-z}$ with $\varepsilon \to +0$



If 3.) $f(u)$ is real for $u \leq a$, Schwarz's reflection principle gives $f(z^*) = f^*(z)$

so $f(z) = \frac{1}{\pi}\int_a^\infty du\, \frac{f_1(u)}{u-z}$

where $f_1(u) = \frac{1}{2i}\left(f(u+i\varepsilon) - f(u-i\varepsilon)\right) = \mathrm{Im}\, f(u)$

is assumed finite

from $F(z) = \frac{1}{\pi}\int_0^\infty du \, \frac{\operatorname{Im} F(u)}{u-z}$.

we can obtain $\operatorname{Re} f(x) = \frac{1}{\pi} P\int_0^\infty du \, \frac{\operatorname{Im} f(u)}{u-x}$

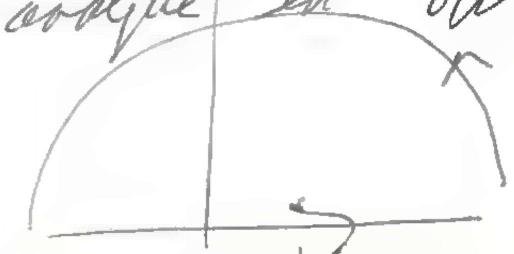
when $x \to x+i\epsilon$.

and $\operatorname{Im} F(x) = \frac{\pi}{\pi} \operatorname{Im} f(x)$ as required for consistency.

Goldberger gives example of forward scattering of photons by protons

$F(q)$ is forward scattering amplitude as function of photon energy q.

Then $f(q)$ is analytic in upper half-plane



then $f(q)$

$F(q) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dq' \, \frac{f(q')}{q'-q}$ $\quad$ q approaches real axis from above.

It follows that

$$f(q) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} dq' \, \frac{f(q')}{q'-q} .$$

cf Gasio p. 350

then

$$\mathrm{Re}\, f(q) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dq' \, \frac{\mathrm{Im}\, f(q')}{q'-q} .$$

using $\mathrm{Im}\, f(-q') = -\mathrm{Im}\, f(q')$

we can write

$$\mathrm{Re}\, f(q) = \frac{1}{\pi} P \int_{0}^{\infty} 2q' dq' \, \frac{\mathrm{Im}\, f(q')}{q'^2-q^2} .$$

This was one of the first examples of a dispersion relation to be derived by Gell-Mann, Goldberger & Thirring in 1955.

Gasio. then goes on to use the Mandelstam rep to obtain forward scattering dispersion relations in pion-nucleon scattering, and other reduction formulae to hold.

incident wave

$\ell = 0$

$$(2ikr)\psi \propto e^{ikr} - e^{-ikr} + \left(e^{2i\eta_\ell} - 1\right) e^{ikr}$$

$$= -e^{-ikr} + e^{2i\eta_\ell}\, e^{ikr}$$

$$= -\left(e^{-ikr} - S\, e^{ikr}\right)$$

Partial waves; dispersion relations: Regge theory [1]

we adopt the partial wave expansion

$$f(s,t) = \sum_{n=0}^{\infty} (2n+1) \cancel{a_n P_n} \; a_n(s)\, P_n(\cos\theta) \quad (1)$$

where $a_\ell = \frac{1}{2ik}\left(e^{2i\eta_\ell} - 1\right) = \frac{e^{i\eta_\ell} \sin\eta_\ell}{k}$

where $k^2 \propto E$, energy of particle in potential scattering.

In general $e^{2i\eta_\ell}$ is partial-wave coeff. for S-matrix

$\left(e^{2i\eta_\ell} - 1\right) \propto$ coeff for scattering amplitude. $S = 1 + f$ ordinarily.

[ordinarily $e^{-ikr} - S e^{ikr}$ for $\ell=0$, cf. NST p. 391]

Thus $\psi \propto e^{ikr} \cancel{-} e^{-ikr} + f \cdot e^{ikr}$

$= \cancel{-} e^{-ikr} + (1+f) e^{ikr} = \cancel{-} e^{-ikr} + S e^{ikr}$

shows general relationship between f & S.

$\left[\psi \propto -e^{-ikr} + S e^{ikr} = -\left(e^{-ikr} - S e^{ikr}\right) \right]$

Then from (1) $a_n = \frac{1}{2} \int_{-1}^{1} d(\cos\theta)\, \tilde{f}(s, \cos\theta)\, P_\ell(\cos\theta)$

since $\int_{-1}^{1} P_\ell^2(u)\, du = \frac{2}{2\ell+1}$ and $\tilde{f}(s, \cos\theta) = f(s,t)$

from, using unitarity condition for elastic scattering yields. in terms of f or "T" notation and similar.

we obtain $i(a_e^* - a_e) = 2|a_e|^2$.

$$\text{whence } a_e = \frac{e^{2i\eta_e}-1}{2ik} = e^{i\eta_e}\frac{(e^{i\eta_e}-e^{-i\eta_e})}{2ik}$$

η_e is real. with g in elastic region

$$= \frac{e^{i\eta_e}\sin\eta_e}{k}.$$

k is just a real constant here.

If inelastic processes also occur, we write.

$$i(a_e^* - a_e) = 2|a_e|^2 + 2\rho_e.$$

$S_e(\text{elastic}) = 1 + 2i\,a_e$. as before.

we notice that $\rho_e = \text{Im}\, a_e - |a_e|^2$. — (1)

we can infer $0 \leq \rho_e \leq 1/4$.

Writing $a_e(W) = \dfrac{\eta_e e^{2i\eta_e} - 1}{2i}$ (change in notation).

we can choose η_e real, in range $0 \leq \eta_e \leq 1$

then $\rho_e = 1/4(1-\eta_e^2)$

It follows since from ①. 3

$$\theta_e = \frac{(\eta_e \cos 2\delta_e - 1) + i\,\eta_e \sin 2\delta_e}{2i}$$

$$\mathcal{I}_e = -\tfrac{1}{2}(\eta_e \cos 2\delta_e - 1) - \tfrac{1}{4}\left[(\eta_e \cos 2\delta_e - 1)^2 + \eta_e^2 \sin^2 2\delta_e\right]$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\eta_e \cos 2\delta_e - \tfrac{1}{4}(\eta_e^2 + 1 - 2\eta_e \cos 2\delta_e)$$

$$= \tfrac{1}{4}(1 - \eta_e^2) \text{ as stated.}$$

We can define 3 partial cross-sections:—

$\sigma^l(\text{elastic})$ $\sigma^l(\text{inelastic})$ $\sigma^l(\text{total})$, such that

$$\sigma(\text{total}) = \sum_{l=0}^{\infty} \sigma^l(\text{total})$$

σ_{total} is still given by the optical theorem.

$$\propto \sum_{l=0}^{\infty} (2l+1)\, \text{Im}\, a_l \, P_l(1)$$

whence we find

$$\left.\begin{aligned}
\sigma^l(\text{total}) &= \frac{(2l+1)\pi}{k^2}(2 - 2\eta_e \cos 2\delta_e) \\
\sigma^l(\text{elastic}) &= (2l+1)\pi/k^2\,(1 - 2\eta_e \cos 2\delta_e + \eta_e^2) \\
\sigma^l(\text{inelastic}) &= \frac{(2l+1)\pi}{k^2}(1 - \eta_e^2)
\end{aligned}\right\}
\begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

so that ① = ② + ③.

Optical theorem from Faxen Holtsmark formula

Start from $f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\eta_l}-1)P_l(\cos\theta)$

which is derived from plane wave expansion.

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)\, i^l j_l(kr) P_l(\cos\theta)$$

$$j_l = \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr).$$

$$j_l \sim (kr)^{-1}\sin(kr - \tfrac{1}{2}l\pi)$$

or $f(\theta) = \frac{1}{k}\sum(2l+1)e^{i\eta_l}(\sin\eta_l)P_l(\cos\theta)$

Then $\mathrm{Im}\, f(0) = \frac{1}{k}\sum(2l+1)\sin^2\eta_l$

But $\sigma_{total} = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{k^2}\sum(2l+1)\sin^2\eta_l$

$$= \frac{4\pi}{k}\,\mathrm{Im}\, f(0)$$

This is still true for inelastic scattering

In presence of inelastic scattering we can write

$$\sigma^{l}_{sc.} \sim r^{-1} C_l\, e^{ikr} P_l(\cos\theta), \quad C_l = e^{2i\eta_l}-1 \text{ for elastic scattering}$$

then $\sigma_l(\text{elastic}) \propto |C_l|^2$

and $\sigma_l(\text{total}) \propto \mathrm{Im}\, C_l$, from optical theorem

N.B. change in alignment
of the title

then we find σ_e(inelastic) = σ(total) − σ(elastic) 5

Alternatively we can compute σ(inelastic) by calculating nett outward radial flux across large sphere using $\psi = \psi^l_{inc} + \psi^l_{scattered}$

where $N = -\frac{c\hbar}{2\mu}\int\left(\psi^* \frac{\partial \psi}{\partial r} - \frac{\partial \psi^*}{\partial r}\cdot\psi\right) dS$.

and have find

$$\sigma^l_{reaction} = \frac{4\pi}{2l+1}\left\{\frac{i}{2k}(2l+1)(C_l^* - C_l) - |C_l|^2\right\}$$

↑ σ(total) from optical theorem ↑ σ elastic

Another way of stating problem is to use a complex phase shift

i.e. put $C_l = \frac{1}{2ik}\left(e^{2i\eta_l} - 1\right)$ with $\eta_l = \lambda_l + i\mu_l$ earlier

then $C_l = \frac{1}{2ik}(2l+1)\left(e^{2i\lambda_l}e^{-2\mu_l} - 1\right)$

(so in previous notation we are taking $\eta_l = e^{-2\mu_l}$)

& then put (1) (2) & (3) as above.

e.g. $\sigma_{\substack{\cancel{\text{rotation}}\\ \text{reaction}}} = \frac{(2\ell+1)\pi}{k^2}\left(1-\eta_\ell^2\right)$

$$= \frac{(2\ell+1)\pi}{k^2}\left(1-e^{-4\mu\ell}\right)$$

$$= \frac{(2\ell+1)\pi}{k^2}\, e^{-2\mu\ell}\left(e^{2\mu\ell}-e^{-2\mu\ell}\right)$$

$$= \frac{2\pi}{k^2}\cdot(2\ell+1)\, e^{-2\mu\ell}\sinh 2\mu\ell$$

as given in $\Pi\ni\Pi\delta R$

Elton p. 151

We have $f_\ell = \dfrac{e^{2i\eta_\ell} - 1}{2i}$ Reactance Notation

$S_\ell = \cancel{1+f_\ell} = 1 + 2if_\ell = e^{2i\eta_\ell}$.

write $R_\ell = \tan\eta_\ell$.

$S_\ell = e^{2i\eta_\ell}$

S = scattering matrix

$\psi_\ell \propto e^{-ikr} - S_\ell e^{ikr}$

relates R_ℓ & S_ℓ or $\propto \sin kr + \tan\eta_\ell \cos kr$.

in simple case of potential scattering $= \sin kr + R_\ell \cos kr$.

R = reactance matrix.

then. $S_\ell = \dfrac{1 + iR_\ell}{1 - iR_\ell}$

if R is Hermitian, S is unitary.

$$S_\ell = \frac{1 + i\tan\eta_\ell}{1 - i\tan\eta_\ell} = \frac{(1 + i\tan\eta_\ell)(1 + i\tan\eta_\ell)}{1 + \tan^2\eta_\ell}$$

$$= \frac{1 - \tan^2\eta_\ell}{1 + \tan^2\eta_\ell} + 2i\,\frac{\tan\eta_\ell}{1 + \tan^2\eta_\ell}$$

these results can be generalized for multi-channel scattering (cf M & M)

$$= \cos 2\eta_\ell + 2i\,\frac{\sin\eta_\ell}{\cos\eta_\ell \sec^2\eta_\ell}$$

$$= \cos 2\eta_\ell + 2i\sin\eta_\ell\cos\eta_\ell$$

$$= \cos 2\eta_\ell + 2i\sin\eta_\ell\cos\eta_\ell = \cos 2\eta_\ell + i\sin 2\eta_\ell$$

$$= e^{2i\eta_\ell}$$

by de Moivre's theorem

According to M & M p. 133.

General definition of partial wave S-matrix S_l

In general we write

$$G_0(0)=0 \qquad G_l \sim A(k)\sin(kr - \tfrac{1}{2}l\pi + \eta_l)$$

we introduce $f_l(\pm) \sim e^{\mp ikr + \frac{1}{2}il\pi}$

then with $A(k) = e^{i\eta_l}$

$$G_l \to e^{i\eta_l}\sin(kr - \tfrac{1}{2}l\pi + \eta_l)$$

$$\sim e^{i\eta_l}\,\frac{e^{ikr - \frac{1}{2}il\pi + i\eta_l} - e^{-ikr + \frac{1}{2}il\pi - i\eta_l}}{2i} \qquad (1)$$

also $G_l = \frac{1}{2}i\left[f(+) + (-1)^{l+1}S_l\, f(-)\right\} \qquad S_l = e^{2i\eta_l}$

$$\sim \tfrac{1}{2}i\left[e^{-ikr + \frac{1}{2}il\pi} + (-1)^{l+1}e^{2i\eta_l}e^{ikr + \frac{1}{2}il\pi}\right] \qquad (2)$$

(1) gives $\frac{1}{2}i\, e^{-ikr + \frac{1}{2}il\pi} - e^{2i\eta_l}\cdot e^{ikr}\cdot e^{-\frac{1}{2}il\pi}$

So we require for (1) & (2) to agree

$$(-1)^{l+1}e^{\frac{1}{2}il\pi} = -e^{-\frac{1}{2}il\pi}$$

or $(-1)^{l+1} = -e^{-il\pi} = (-1)^l * (-1) = (-1)^{l+1}$, which is correct.

for a general

$$\psi_\ell \propto e^{-ikr} + (-1)^{\ell+1} e^{2i\eta_\ell} \cdot e^{ikr}$$

for $\ell = 0$

$$\psi \propto e^{-ikr} - e^{2i\eta_\ell} e^{ikr}$$

So general for odd ℓ

$$\psi_\ell \sim e^{-ikr} + S_\ell e^{ikr}$$

for even ℓ $\psi_\ell \sim e^{-ikr} - S_\ell e^{ikr}$.

General definition has been given by Kohn which includes both forms.

Writes $G \sim \left(A e^{-i(kr - \frac{1}{2}\ell\pi)} - B e^{i(kr - \frac{1}{2}\ell\pi)}\right)$

then $B = SA$ defines S - relation

i.e. $G \sim A\left\{e^{-i(kr - \frac{1}{2}\ell\pi)} - S_\ell e^{i(kr - \frac{1}{2}\ell\pi)}\right\}$ (2)

as general definition

Notice change in sign of $e^{\frac{1}{2}i\ell\pi}$ between A & B terms

P.T.O.

Similarly for ℓ. notion, take radial functions

$$G \simeq \left(C \sin(kr - \tfrac{1}{2}\ell\pi) + D \cos(kr - \tfrac{1}{2}\ell\pi) \right)$$

then $D = R_\ell C$ defines Reaction Matrix R

so in terms of R

$$G \simeq C \left(\sin(kr - \tfrac{1}{2}\ell\pi) + R_\ell \cos(kr - \tfrac{1}{2}\ell\pi) \right) \quad \text{cf. } (3)$$

notes + sign in (3) contrasted with − sign in (2)

Partial Wave Dispersion Relations

Derived from Mandelstam Representation multiplied by $P_\ell(\cos\theta)$ & integrated over ~~~~ θ from 0 to π — $\cos\theta$ expressed in terms of s and t, leads to cuts in the partial wave amplitude, partly derived from θ integration

This in turn leads to dispersion relations for $a_\ell(s)$ in the complex variable s

On R.H. cut disc. $a_\ell(s)$ is $2i\,\mathrm{Im}\, a_\ell(s)$ on physical sheet
$\propto |a_\ell(s)|^2$

on L.H. cut we do not know what disc. $a_\ell(s)$ is at all (no longer simply related to crossed partial wave amplitude, since unitarity is now simple, but crossing is complicated)

L.H. disc $a_\ell(s)$ plays role of potential

L.H. cut is referred to as the non-physical cut

thus we start here. for pion-pion scattering

$t = -2q^2(1-\cos\theta) \quad s = 4(q^2+m_\pi)^2$

so $\cos\theta = \left(1 + \frac{2t}{s-4m_\pi^2}\right)$

we write for scattering amplitude

$$T = \frac{1}{\pi^2}\int_{4m_\pi^2}^{\infty}\int dx\,dy\,\rho(x,y)$$

$$\times\left[\frac{1}{(x-s)(y-t)} + \frac{1}{(x-t)(y-u)} + \frac{1}{(x-u)(y-s)}\right]$$

; then express t & u in terms of q^2 & θ

to get the partial-wave amplitude

$$b_\ell(u) \propto \iint dx\,dy\,\rho(x,y)\left((1+(-1)^\ell\right)$$

$$\times\left(\frac{1}{x-s} + \frac{1}{x+y+4q^2}\right)Q_\ell\left(1+\frac{y}{2q^2}\right)$$

where Q_ℓ is Legendre function of 2nd kind, defined with cut from -1 to 1

Note on

Resonances, virtual states, bound states

Note Pole in scattering amplitude at $k = a - ib$ $\substack{a>0 \\ b>0}$

leads to $E \propto k^2 = a^2 + b^2 - 2aib$

In $e^{-iEt} \to e^{-2abt} \times$ Resonance term

connected to decay of unstable particle

Resonance near $k = a$, if b is small

r dependence is $e^{ikr} = e^{br} \times$ Resonance term

same as for theory of α - decay

For $a = 0$ — r dependence is e^{br}, leads to virtual state

For $a = 0$, $b < 0$ r-dependence is e^{-br}

— true bound state.

Writing $t = \sqrt{s}\, f_\ell$ $\quad \nu = q^2$; $f = f(\nu)$

then we obtain dispersion relation.

$$f_\ell = \frac{1}{\pi}\int_0^\infty d\nu' \frac{\rho_\ell(\nu')|f_\ell(\nu')|^2}{\nu' - \nu - i\varepsilon} + \frac{1}{\pi}\int_{-D}^{m_\pi^2} d\nu' \frac{\phi_\ell(\nu')}{\nu' - \nu - i\varepsilon}$$

for physical amplitude
where $\mathrm{Im}\, f_\ell(\nu) = \rho_\ell(\nu)|f_\ell(\nu)|^2$ and
p.v. cal.

Effective range approx write $\nu^\ell \mathrm{Re}\frac{1}{f_\ell(\nu)} \approx a_\ell + b_\ell \nu$

leads to $\frac{q^{2\ell+1}}{\sqrt{s}} \cot \delta_\ell(\nu) = a_\ell + b_\ell \nu$.

and $f_\ell(\nu) = \frac{1}{q\cot\delta_\ell(\nu) - iq}$ $\left(= \frac{e^{i\delta_\ell}\sin\delta_\ell}{q} \right)$

which is contained here

Effective range expansion is of $q\cot\delta_\ell$ in powers of $q(\text{or } \nu)$

leads to B-W formula of the resonance form near zero of $q\cot\delta_\ell$

& the bound state near zero of $q\cot\delta_\ell - iq$.

So leads to theory of resonance; bound states
near $k^2 = 0$

(i) can be solved if the above — 5 —

— Key role to potential, played

by N/D method f.g.

Bootstrap approach ~~resonance in a~~

The dynamics from scattering in crossed

channel. If resonance in u-channel

is exchanged to "produce" the, then

will also lead to same resonance

in S-channel — e.g. ρ bootstrap

2 pions exchange ρ-meson, leads

to ρ-meson resonance e.g.

(1) Opp. may best be written

$$F(s, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell(\cos\theta)$$

Residue of $\operatorname{cosec} \pi\ell = R_\ell$

Res at $\ell\pi$ $\frac{1}{R_\ell} = \lim_{\theta \to \ell\pi} \frac{(\theta - \ell\pi)}{\sin \theta} = \lim_{x \to 0} \frac{x}{\sin(\theta - x)}$ $x = \theta - \ell\pi$

$x = \theta - \ell$

$$\text{Res} \lim_{\substack{\theta \to \ell \\ \text{an integer}}} \frac{(\theta - \ell)}{\sin \pi\theta} = \lim_{x \to 0} \frac{x}{\sin \pi(x+\ell)}$$

$$= \lim_{x \to 0} \frac{x}{\sin \pi x \cos \pi\ell} = (-1)^{\ell} \lim_{x \to 0} \frac{x}{\sin \pi x}$$

$$= \frac{(-1)^{\ell}}{\pi} \text{ as stated}$$

Regge Poles.

We start from scattering amplitude in form

$$F(w, \cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, a_l(w)\, P_l(\cos\theta) \quad (1)$$

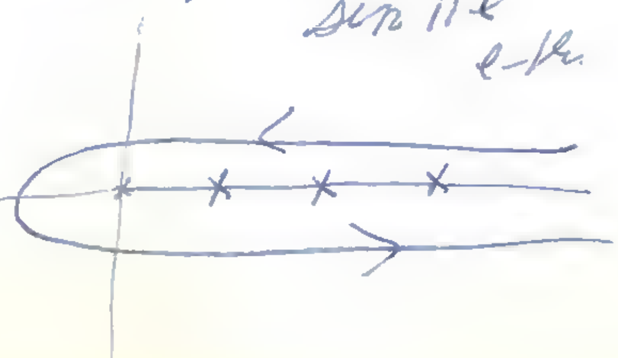
and seek, first of all, an interpolating function for $a_l(s)$ — $\sqrt{s}$ is energy in the c.m. frame.

$a(l, s) = a_l(s) \quad l = 0, 1, 2 \ldots$

uniqueness of $a(l,s)$ guaranteed by Carlson's theorem (assuming $a(l,s)$ vanishes for large l in some manner … cone …)

Then express Σ as an integral over dl along an appropriate contour, by introducing $\operatorname{cosec} \pi l$ or a meromorphic function with residue $\frac{(-1)^l}{\pi}$ at poles at $l = 0, 1, 2 \ldots$

Hence $f(w, \cos\theta) = \frac{1}{2i} \int_C dl \cdot (2l+1)\, a(l, s) \frac{P_l(-\cos\theta)}{\sin \pi l}$



C is enclosing the

poles of integrand at $l = 0, 1, 2 \ldots$

poles of $a(l, s)$ at Regge poles.

Background integral is along line $\mathrm{Re}\, \ell = -\frac{1}{2}$.

<u>Regge assumption</u> is that to right of $\mathrm{Re}\, \ell = -\frac{1}{2}$ the function $a(\ell, s)$ has only isolated poles.

Watson - Sommerfeld Transform:

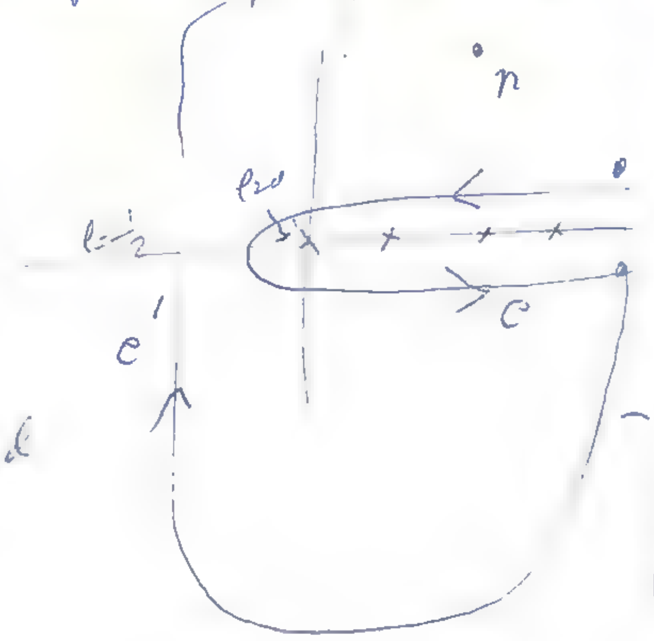
we now deform path of ~~[illegible]~~ integration thus:

ℓ-plane

Note $s = 4(m^2+q^2)$ for $q^2=0$ we have $s=4m^2$

q is momentum in c.m. frame

$\sqrt{s}$ is ~~energy~~ in s-channel in c.m. frame.



$q^2>0$ all poles in complex plane — poles on real axis only for $q^2<0$

Then $\frac{1}{2i}\int_{C'} + \frac{1}{2i}\int_C = -\pi \sum_n \alpha_n(s)$ α_n is residue at n^{th} Regge pole.

If n' is Regge pole on real axis

then $\pi \sum_{n'} \alpha_{n'}(s) + f + \frac{1}{2i}\int_{C'} = -\pi \sum_n \alpha_n(s)$

or $f = -\pi \sum_n \alpha_n(s) - \pi \sum_{n'} \alpha_{n'}(s) - \frac{1}{2i}\int_{C'}$

$= -\pi \sum_n \alpha_n(s) - \frac{1}{2i}\int_{C'}$ —(1)

where n now includes all the Regge poles

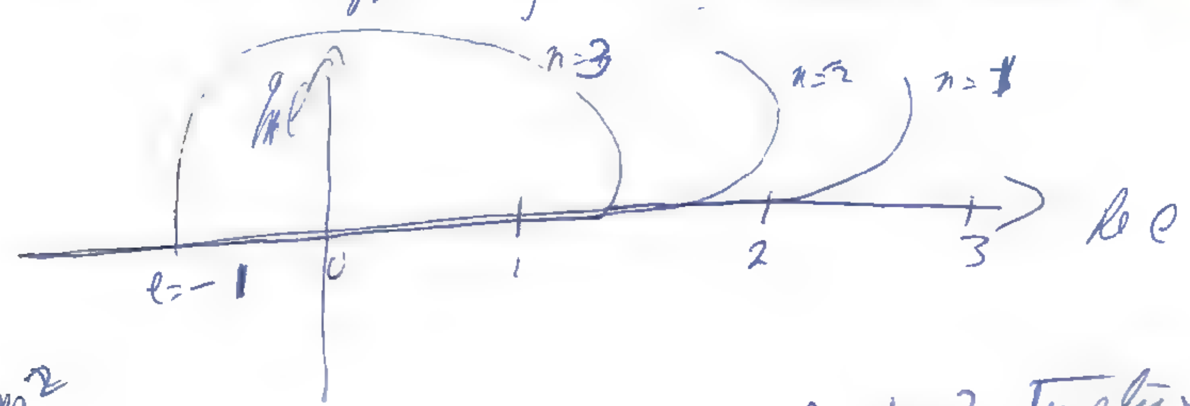
$-\int_{C'}$ is the so-called background integral

n^{th} Regge-pole is at $l = l_n(s)$

plot of complex no $l_n(s)$, parametrized by s is called a Regge Trajectory

Alternative definition is $\mathrm{Re}\, l_n(s)$ against $\mathrm{Re}\, s$

Example of Regge trajectory for Yukawa potential



$s < 4m^2$ $S < 0$, $l_n(s)$ on real axis, $s > 4m^2$ trajectory leaves real axis

value of s ($< 4m^2$) for which $l_n(s) = 0, 1, 2, 3$

give the bound states. As strength of potential increases trajectory moves to right, higher, higher poles. generate $l=0$ bound states, $l=1$ also start to appear

at $s = -\infty$ & $+\infty$ trajectory goes to $l=-1$

Returning to (1) we change notation and write

$$a(\ell,s) \simeq \frac{\beta_n(s)}{\ell - \alpha_n(s)} \quad \text{near } n^{th} \text{ pole}$$

then β_n is residue.

Regge trajectory is then a plot of $\alpha_n(s)$ parametrized by s or a plot of $\mathrm{Re}\,\alpha_n$ against s (assumed real)

then (1) can be written as

$$f(s,\cos\theta) = -\frac{1}{2i}\int_{C'} d\ell \frac{(2\ell+1)}{\sin\pi\ell} a(\ell,s) P_\ell(-\cos\theta)$$

$$-\pi \sum_n \frac{(2\alpha_n(s)+1)\beta_n(s)}{\sin\pi\alpha_n(s)} P_{\alpha_n(s)}(-\cos\theta)$$

$\cos\theta$ can now become complex, i.e. we can let t be complex — lead to justification for the Mandelstam double dispersion relation, which was original reason for Regge's work

$\longrightarrow$

* We are saying aný result

$$P_n(\mu) = \mu^n + \text{—}$$

i.e. $P_n(x) \to x^n$ for large x

even if n is complex

i.e. $|P_n(x)| \to x^{\mathrm{Re}\, n}$. Where we have assumed x real & n complex.

true as $z = \cos\theta \to \infty$. → Are these correct? *

$$P_\ell(z) \sim |z|^\ell \text{, and for complex } \ell$$

$\mathrm{Re}\,\ell > -\frac{1}{2}$, $|P_\ell(-z)| \sim |z|^{\mathrm{Re}\,\ell}$, as $z \to \infty$

Background integral $\propto |z|^{-1/2}$ as $z \to \infty$

(and hence also $t \to \infty$) Remember $\cos\theta = z$

$$= 1 + \frac{2t}{s-4m^2}$$

Let α_1 be leading Regge pole ie one with largest value of $\mathrm{Re}\,\alpha(s)$,

then α_1 determines behaviour of $f(s, \cos\theta)$ as $\cos\theta \to \infty$ and of $f(s,t)$ as $t \to \infty$.

viz, for one pole:

$$f(s,t) \sim -\frac{\pi(2\alpha_1+1)\beta_1}{\sin\pi\alpha_1}(-z)^{\alpha_1}$$

$$= c_1(s)\, t^{\alpha_1(s)}$$

where $c_1 = -\dfrac{\pi(2\alpha_1+1)\beta_1(s)}{\sin\pi\alpha_1}\left(-\dfrac{2}{s-4m^2}\right)^{\alpha_1}$

1'

In the t - channel, the mean argument 6
on continuation to $s < 0$, so $s \to t$
$t \to s$

by crossing symmetry for spinless bosons

$$f(s,t) \sim c_1(t)\, s^{\alpha_1(t)} \quad \text{—— (1)}$$

for large s

where α_1 is leading pole in the t-channel.

for n - poles, in general

$$f(s,t) \sim \sum_n \left(\frac{s}{s_0}\right)^{\alpha_n(t)} c_n(t)$$

where $\alpha_n(t)$ is location of Regge pole.

which determine bound states &
resonances in the t- channel (where $t > 0$)

~~is not $t < 0$~~

From (1) high energy behaviour comes, cohesively
from $\operatorname{Re}\alpha_1$

Prem (i) alone we have

$f(s,t) \sim c_1(t)\, s^{\alpha_1(t)}$ large s

Assume Pomeron pole α_P dominates

and that $\alpha_P(0) = 1$

then $f(s,t) \sim c_1(t)\, s^{\alpha_P(t)}$

Th cross-section is given by

of $\sigma_{total} \propto \frac{1}{q} \operatorname{Im} f(s,0)$ $\quad s \propto q^2$, high energ.

and $\frac{d\sigma}{dt} \propto \left|\frac{f(s,t)}{q^2}\right|^2 = \frac{1}{q^2}|f(s,t)|^2 = \frac{1}{s}|f|^2.$

$\therefore \frac{d\sigma}{dt} \propto \frac{1}{s} \cdot |c_1(t)|^2\, s^{2\operatorname{Re}\alpha_P(t)} \simeq \frac{1}{s}|c_1(0)|^2 s^{2\operatorname{Re}\alpha_P t}.$

so $\left(\frac{d\sigma}{dt}\right)_{t=0} = \frac{1}{s}|c_1(0)|^2 s^{2\operatorname{Re}\alpha_P(0)}$

$\therefore \frac{1}{s}|c_1(0)|^2 = \left(\frac{d\sigma}{dt}\right)_{t=0} s^{-2\operatorname{Re}\alpha_P(0)}$

we have $\frac{d\sigma}{dt} = \left(\frac{d\sigma}{dt}\right)_{t=0} s^{2\operatorname{Re}(\alpha_P(t)-\alpha_P(0))}$

$= \left(\frac{d\sigma}{dt}\right)_{t=0} e^{2(\alpha_P(t)-1)\ln s}$

s is measured, more generally in terms of a parameter s_0

so that we can write -

$$\frac{d\sigma}{dt} = \left(\frac{d\sigma}{dt}\right)_{t=0} e^{2(\alpha_P(t)-1)\ln(s/s_0)}$$

for small t $\quad \alpha_P(t) = 1 + \varepsilon_P t$

$\varepsilon_P = \alpha_P'(t)\big|_{t=0}$

and $\frac{d\sigma}{dt} = \frac{d\sigma}{dt}\Big|_{t=0} e^{2\varepsilon_P . t \ln(s/s_0)}$

which is of exponential form for small t $\quad \left(\frac{d\sigma}{dt}\Big|_{\text{exponential}} = \frac{d\sigma}{dt}\Big|_{t=0} e^{at+bt^2}\right)$;

ε_P is positive

so that width a increases with s logarithmically, (remember t is -ve and $\alpha_P(t)$ is then real, by analogy with potential scattering).

Here we have a shrinkage of diffraction peak as energy increases, not observed - perhaps present experimental energies not high enough for simple pole approximation to be valid?

$\alpha_P(0) = 1$ then that $\sigma_{total} \to$ (constant) $\cancel{\text{cobalt}}$ $\cancel{B}$

$$= \cancel{\frac{1}{q} \operatorname{Im} e(t)}$$

$$= \frac{1}{q} \operatorname{Im} c_1(0) \cdot \times s$$

$c_1 \propto \frac{1}{q}$ Hence. $\sigma_{tot} \propto c_1(0)$,

and is independent of s.

But pomeron is not a true particle since it has even signature, which is not available to $l = 1$, in relativistic theory.

<u>Gribov - Froissart Relativistic treatment</u>

showed by considering partial wave expansion derived from Mandelstam representation (as above) that $a_l(s)$ does not satisfy Carlson's condition — instead defined two amplitudes $a^+(l, s)$ and $a^-(l, s)$ such that

$$a^+(l, s) = a_l(s) \quad l = 0, 2, 4 \ldots$$

$$a^-(l, s) = a_l(s) \quad l = 1, 3, 5 \ldots$$

Here a^+ and a^- can be used for the unique extrapolation as they satisfy conditions for applying Carlson's theorem

A pole of $a^+(l,s)$ for l = even l s.t. $s < 4m^2$ corresponds to a bound state, but a pole of $a^+(l,s)$ for l odd does not, hence pole is not a bound state, (or true particle)

<u>Bound states poles & resonances</u>

From Regge form for $f(s,t)$ near integral l for pole with $s < 4m^2$ we get a pole in f which corresponds to a bound state

which for $s > 4m^2$ if $\alpha_n(s) = l + \text{Im}\,\alpha_n(s)$ where $\text{Im}\,\alpha_n(s)$ is small and $l = \text{Re}\,\alpha_n(s)$ we obtain a resonance form for $\text{Re}(s)$ as derived in Gono. p <u>485</u>

coincident singularities

If $F(x) = 0$ has roots at $x_1, x_2 \ldots$

double root at $x = x_2$ means F has form (for polynomial) $F(x) = (x-x_2)^2 \phi(x)$

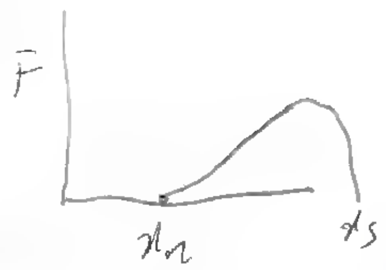
$$\left.\begin{aligned} F(x_2) &= 0 \\ \& \quad F'(x_2) &= 0 \end{aligned}\right\} \quad \text{also follows from} \quad F(x_3) \simeq F(x_2) + \frac{\partial F}{\partial x_2} \cdot (x_3 - x_2)$$

for x_3 near to x_2 but not coincident.

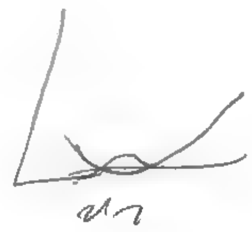
means $0 \simeq 0 + \frac{\partial F}{\partial x_2} (\delta x_2)$

is true only if $\frac{\partial F}{\partial x_2} = 0$.

In general.



near



— i.e. no curve near is a tangent to the curve at a double zero

i.e. coincident zero is at condition for F to be a minimum or zero.

but a boundary.

Queries for Carlitz

(1)

① In Gas. incorrect to say we must use
$\frac{\partial A_\mu^{(+)}}{\partial x}|\Phi\rangle = 0$ as subsidiary condition

② Definition of S-matrix

Two operators are defined S & V such that

$(\phi_i; S\,\phi_j) = (\psi_i^{in}, V, \psi_j^{in}) = S_{\alpha\beta}$ where ϕ are eigenfcts of H_0 & ψ_i are eigenfcts of H $= \langle \psi_i^{out} | \psi_j^{in} \rangle$ ~~$\langle \psi_{in} | \psi_j^{out} \rangle$~~

Then S-matrix element is $S_{\alpha\beta}$. $= \langle \psi_i^{out} | \psi_j^{in} \rangle$ ~~$\langle \psi_{in} | \psi_{out} \rangle$~~

where $S = U(\infty, -\infty)$, But $V \neq S$

& V is used in Yang-Feldman review

$$\phi^{out} = V^{-1} \phi^{in} V$$

But, according to Jauch, Rohrlich

V & S are identical

where is the discrepancy?

a.) Schweber uses interaction rep. at $t = 0$

b.) J & R use interaction rep at $t = -\infty$, and assume identity of $\phi_{interaction}$ & ϕ_{free} in Y-F. formalism.

(3) Discuss L.S.Z. reduction for
$$H' = \underbrace{U}_{\text{strong}} + \underbrace{V}_{\text{weak}}.$$

(4) In unitary symmetry F in $[F, \phi] = i f_i a \ldots$

F is generator of infinitesimal unitary transformation i.e. $\mathcal{F}^{(i)} = \int \Pi(x) F^{(i)} \psi(x) d^3x$.
where. $F^{(i)}$ are the infinitesimal matrices.
$\mathcal{F}^{(i)}$ satisfy the same commutation relations as the $F^{(i)}$ and provide a representation of same.
i.e. $\mathcal{F}^{(i)} = \int \Pi_i F^{(i)}_{,k} \psi_k \, d^3x$.

(5) in Cabibbo's current hypothesis (cf. Sakata stem function)
weak stru current from all 8 nuclear ... are we abandoning Pupp-Gellmann ... — of weak strangeness-changing coupling is between $p \Lambda$.

Answer to Q. 2

By Schwinger $V = \Omega^{(+)} \Omega^{(-)-1} = U(0,-\infty)\, U(\infty, 0)$ (1)

$$\Omega^{(+)} = U(0,-\infty)$$
$$\Omega^{(-)} = U(0, \infty)$$

and $S = U(\infty, -\infty)$

But by J & R. p. 118.

$U(\tau_0, \tau)\, U(\tau_1, \tau_0)$ is independent of τ_0.

Take $\tau = -\infty$
$\tau_1 = +\infty$

then $U(\tau_0, -\infty)\, U(\infty, \tau_0)$ is independent of τ_0

$= U(\infty, -\infty)$ with $\tau_0 = -\infty$

$= U(0,-\infty)\, U(\infty, 0)$ with $\tau_0 = 0$

$= V$

Hence $S = V$ as stated in J & R and discrepancy is explained.

Note (1) follows from: $\langle \psi^{in} | V | \psi^{in} \rangle = \langle \phi^{in} | S | \phi^{in} \rangle$
and use $|\psi^{in}\rangle = U(0,\infty)|\phi^{in}\rangle$
to give $\langle \psi^{in} | V | \psi^{in} \rangle = \langle \phi^{in} | U(-\infty, 0)\, V\, U(0,-\infty) | \phi^{in} \rangle$
when $S = U(\infty, -\infty) = U(\infty, 0) V U(0,-\infty) = U(-\infty, 0) V U(0, -\infty)$ whence result follows.

(6) in crossed reactions do we go from p_1, p_2 to $p_3\, p_4$ as incident momenta or leave incident momenta at p_1, p_2 as in our text ?

corresponds to : <u>u-channel</u> $p_3 \to -p_3$
$p_1 \to -p_1$

$$\text{So } u \to (p_4+p_1)^2 =$$ c.m. energy^2 in the crossed collision

<u>t-channel</u> $p_4 \to -p_4$
$p_1 \to -p_1$

$$t \to (p_3+p_1)^2 =$$ c.m. energy^2 in the crossed collision

this method is probably the correct one.